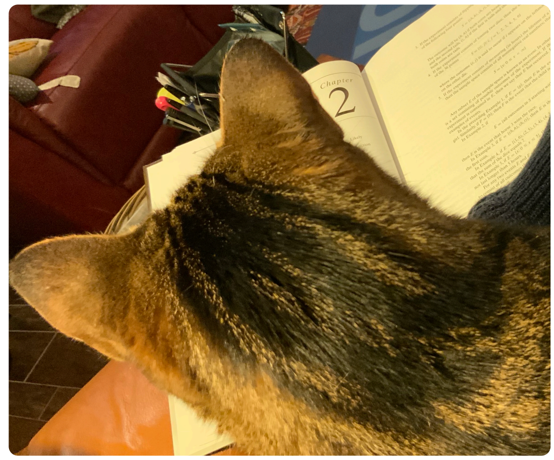
Stats 2D03 - Axioms of Probability.

Sample Spaces and Events



Schrödinger reading about the axioms of probability.

Recall that a random Experiment is an experiment whose out come is not predictable, for example flipping a coin or rolling a die or checking the temperature.

Given a random experiment, the sample space, S, is the set of all possible outcomes.

For example:

a) Flipping a coin:
$$S = \{H, T\}$$

b) Rolling a die:
$$S = \{1, 2, 3, 4, 5, 6\}$$

c) Checking the temperature (in kelvin):
$$S = \{x \geq 0\}.$$

d) Flipping one coin and then flipping another coin:

$$S = \{(h,h), (h,t), (t,h), (t,t)\}. \quad \big(|S| = 4\big)$$

e) Flipping two indistinguishable coins at the same time:

$$S = \{\{h,h\}, \{h,t\}, \{t,t\}\}$$

Sometimes, we are interested in outcomes or sets of outcomes which may be related some how. For example, if we flip three coins consecutively, we may be interested only in the outcomes with exactly two tails.

A subset of a sample space,

$$E \subseteq S,$$

is called an <u>event</u>. If an outcome of an experiment is in an event E, we say that E has occurred.

We have some basic operations for events:

1) <u>If</u> $E, F \subseteq S$ are events, then the union

$$E \cup F = \{x : x \in E \text{ \underline{or} } x \in F\}$$

is an event.

2) If E and F are events, then the intersection

$$EF = \{x : x \in E \text{ \underline{and} } x \in F\}$$

is an event.

<u>N.B.</u>: Typically people write $E \cap F$ for intersection. The textbook for this course writes EF.

3) <u>Complement</u>: If $E \subseteq S$ is an event, then

$$E^c = \{x \in S : x \notin E\}$$ is an event.

We say that two events E and F are <u>mutually exclusive</u> if $EF = \emptyset$.

On the other hand if every event in E is also in F, we say/write $E \subseteq F$.

Note that S and $\emptyset$ are themselves events.

The operations on events follow certain rules/laws:

1) Commutativity: $E \cup F = F \cup E$
$EF = FE$

2) Associativity: $(E \cup F) \cup G = E \cup (F \cup G)$
$(EF)G = E(FG)$

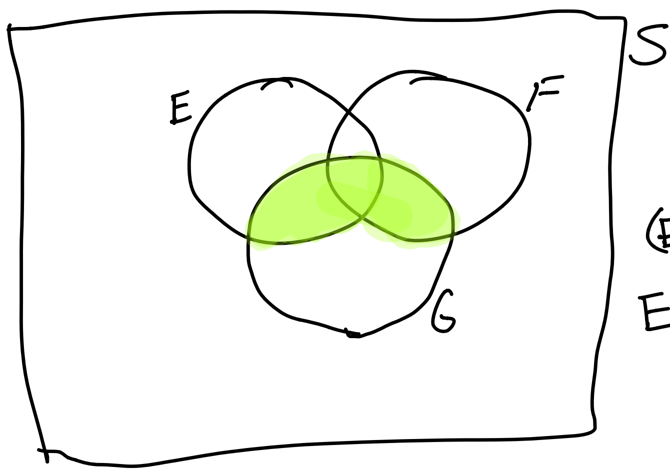
3) Distributivity:

$$(E \cup F)G = EG \cup FG$$

$$(EF) \cup G = (E \cup G)(F \cup G).$$

4) $(E^c)^c = E$.

Venn Diagrams are often useful for visualizing these relationships; for example distributivity.



The shaded region is both $(E \cup F)G$ and also $EG \cup FG$.

We also have the useful "De Morgan's Laws":

$$(E \cup F)^c = E^c F^c$$

$$(EF)^c = E^c \cup F^c.$$

More generally, we have

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c.$$

The Axioms.

Let S be a sample space. Let Σ be the set of events. Probability is a function

$$P : \Sigma \longrightarrow [0,1]$$

satisfying the following axioms:

1) $P(S) = 1$.

2) For any sequence $E_1, E_2, \cdots$ of mutually exclusive events

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

Note that, since S and $\emptyset$ are mutually exclusive, we have that

$$1 = P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset)$$
$$= 1 + P(\emptyset)$$

and hence $P(\emptyset) = 0$.

Ex: Suppose we roll a die.

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Then we have a 100% chance of rolling something and a 0% chance of getting nothing. We have a 50% chance of getting an even number.

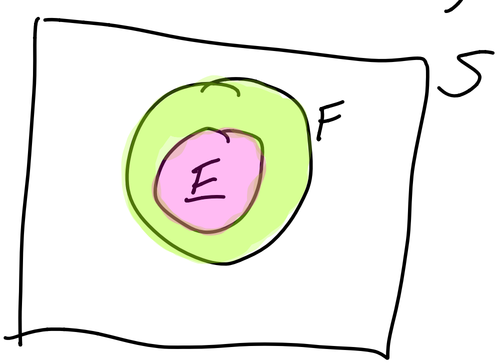
Proposition: For any event E, $P(E^c) = 1 - P(E)$.

Pf: $1 = P(S) = P(E \cup E^c) = P(E) \cup P(E^c)$.

Proposition: $E \subseteq F \Rightarrow P(E) \leq P(F)$.

$$P(F) = P(E \cup E^cF)$$

$$= P(E) + \underbrace{P(E^cF)}_{0 \leq} \geq P(E).$$



$$\left.\begin{array}{c} E \\ E^cF \end{array}\right\} E \cup E^cF = F.$$

Prop: For any events E and F,

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Idea: $P(E \cup F)$ "should" be $P(E) + P(F)$, but we've "doubly counted" the intersection.

Ex: Consider an outcome with three outcomes, $S = \{a, b, c\}$ with each outcome having equal probability $1/3$.

Then $P(\{a,b\}) = 2/3$.

$P(\{b,c\}) = 2/3$

$P(\{b\}) = P(\{a,b\} \cap \{b,c\}) = 1/3$.

$$P(\{a,b\} \cup \{b,c\}) = P(\{a,b,c\}) = 1 = 2/3 + 2/3 - 1/3$$

Pf: In general, $E \cup F = E \,\dot{\cup}\, E^c F$.

Hence $P(E \cup F) = P(E) + P(E^c F)$

Furthermore $P(F) = P(EF) + P(E^c F)$, since $F = EF \cup E^c F$

So $P(E \cup F) = P(E) + P(F) - P(EF)$

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